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# TAYLOR-GÖRTLER LAMINAR INSTABILITY NEAR A CONCAVE SURFACE

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*On the basis of linear and nonlinear approximations, a more accurate value of the critical Görtler number is found and the effect of Taylor–Görtler vortices on surface friction in the transverse direction is determined.*

It is well known [1, 2] that after loss of stability the boundary layer near a concave surface gives rise to the development of secondary flows in the form of a system of Taylor–Görtler eddies with clockwise and counterclockwise rotations whose axes coincide with the direction of the main flow. Görtler was the first to find, on the basis of an analytical approach, that the critical number $Gö_{cr}^{**}$ was equal to 0.58 ($Gö_{cr} = 16$). Subsequently, on the basis of a linear approximation various researchers determined that the value of $Gö_{cr}$ varied from 0.32 to 2.8 (Fig. 1). Such a wide range for $Gö_{cr}$ results from insufficient incorporation of terms in the equations of motion and inadequate accuracy of their solution.

As will be seen below, a linear approach provides a way to determine the local characteristics of a perturbing flow (the change of the coefficient of friction in the lateral direction), but does not permit one to predict the overall increase in the coefficient of friction due to the appearance of Taylor–Görtler vortices. An analysis of Schlichting's solution for flow near a vibrating cylinder [3] shows that this occurs due to the neglect of quadratic terms in the disturbing amplitudes.

In the present work, on the basis of a linear approximation and allowance for virtually all of the terms in the equation of perturbing motion, we determined the most soundly based critical value for the Görtler number and the effect of Taylor–Görtler vortices on the surface friction by taking into account nonlinear effects (quadratic terms).

**Equations of Perturbing Motion (Linear Approximation).** To perform a linear analysis, we use Taylor's relations [2] for the velocity and pressure components (the coefficient of growth of perturbations is considered equal to zero):

$$u = \bar{u} + u_A \cos(\sigma z) \exp\left(\int \gamma dx\right); \quad v = \bar{v} + v_A \cos(\sigma z) \exp\left(\int \gamma dx\right);$$

$$w = w_A \sin(\sigma z) \exp\left(\int \gamma dx\right); \quad p = \bar{p} + p_A \cos(\sigma z) \exp\left(\int \gamma dx\right).$$

Substituting these expressions into the equations of motion and neglecting in the first stage the quadratic terms in the amplitudes of the perturbing quantities, we obtain the equations of the perturbing motion

$$\mathrm{Re}\left[\overset{*}{u}_A \frac{R_w/\delta}{R_w/\delta - \xi} f_\Lambda + v_A D_\Pi \bar{u}^* + \bar{v}^* D_\Pi \overset{*}{u}_A\right] =$$

$$= \bar{v}_{eff}\left[D D_\Pi - \bar{\sigma}^2\right] \overset{*}{u}_A + \left[D + \frac{1}{R_w/\delta - \xi}\right] \overset{*}{u}_A D \bar{v}_{eff} -$$

$$- 2\overset{*}{v}_A \frac{R_w/\delta}{(R_w/\delta - \xi)^2}\left(\frac{\partial \bar{v}_{eff}}{\partial x}\delta\right) + 2\frac{R_w/\delta}{(R_w/\delta - \xi)^3} \bar{v}_{eff} \overset{*}{v}_A \frac{dR_w}{dx}; \tag{1}$$

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$$\mathrm{Re}\left[2\frac{\bar{u}^* u_A^*}{R_w/\delta-\xi}+v_A^* D\bar{v}^*+\bar{v}^* D v_A^*\right]=-Dp_\Pi \mathrm{Re}+\bar{\nu}_{eff}\,[DD_\Pi-\bar{\sigma}^2\,]\,v_A^*+$$

$$+\,2Dv_A^* D\bar{\nu}_{eff}+\frac{R_w/\delta}{R_w/\delta-\xi}\left[D+\frac{1}{R_w/\delta-\xi}\right]u_A^*\left(\frac{\partial\bar{\nu}_{eff}}{\partial x}\delta\right)-\frac{R_w/\delta}{(R_w/\delta-\xi)^3}\bar{\nu}_{eff}\,u_A^*\frac{dR_w}{dx};$$

$$\mathrm{Re}\,\bar{v}^* Dw_A^*=\bar{\sigma}\,p_A^*\,\mathrm{Re}+\bar{\nu}_{eff}\left[D^2-\frac{1}{R_w/\delta-\xi}D-\bar{\sigma}^2\right]w_A-$$

$$-\frac{R_w/\delta}{R_w/\delta-\xi}\left(\frac{\partial\bar{\nu}_{eff}}{\partial x}\delta\right)u_A\bar{\sigma}+D\bar{\nu}_{eff}\,[D_\Pi w_A^*-\bar{\sigma}\,v_A^*\,];\quad D_\Pi v_A^*+\bar{\sigma}\,w_A^*=0\,.$$

These equations permit one to determine in the most complete form the critical conditions for the occurrence of Taylor–Görtler vortices in both laminar and turbulent modes of flow under the effect of various factors such as pressure gradient ($f_\Lambda$) and variable curvature of the surface along its length. Some other factors (injection, suction, external turbulence, etc.) are taken into account through the distribution of the averaged velocities $\bar{u}$, $\bar{v}$ and the effective viscosity, which is the sum of molecular and eddy viscosities.

The system of equations (1) can also be written down in another form if we eliminate from it the pressure $p_A^*$ and the component $w_A^*$:

$$\mathrm{Re}\left[u_A^*\frac{R_w/\delta}{R_w/\delta-\xi}f_\Lambda+v_A^* D_\Pi\bar{u}^*+\bar{v}^* D_\Pi u_A^*\right]=$$

$$=\bar{\nu}_{eff}\,[DD_\Pi-\bar{\sigma}^2\,]\,u_A^*+\left[D+\frac{1}{R_w/\delta-\xi}\right]u_A^* D\bar{\nu}_{eff}-$$

$$-\,2v_A^*\frac{R_w/\delta}{(R_w/\delta-\xi)^2}\left(\frac{\partial\bar{\nu}_{eff}}{\partial x}\delta\right)+2\frac{R_w/\delta}{(R_w/\delta-\xi)^3}\bar{\nu}_{eff}\,v_A^*\frac{dR_w}{dx};\tag{2}$$

$$\mathrm{Re}\left[-2\frac{\bar{u}^*\bar{\sigma}^2}{R_w/\delta-\xi}u_A^*-(\bar{v}^* Dv_A^*+v_A^* D\bar{v})\,\bar{\sigma}^2+D\bar{v}^* D_\Pi Dv_A^*+\bar{v}^* D_\Pi D^2 v_A\right]=$$

$$=\bar{\nu}_{eff}\,[DD_\Pi-\bar{\sigma}^2\,]^2\,v_A^*+D^2\bar{\nu}_{eff}\,[DD_\Pi+\bar{\sigma}^2\,]\,v_A^*+\bar{\sigma}^2\frac{R_w/\delta}{R_w/\delta-\xi}D\times$$

$$\times\left(\delta\frac{\partial\bar{\nu}_{eff}}{\partial x}\right)u_A^*+D\bar{\nu}_{eff}\left[2D^2D_\Pi-\frac{1}{R_w/\delta-\xi}DD_\Pi-\bar{\sigma}^2(D_\Pi+D)\right]v_A^*+\bar{\sigma}^2\frac{R_w/\delta}{(R_w/\delta-\xi)^3}\bar{\nu}_{eff}\,\bar{u}_A\frac{dR_w}{dx}.$$

The systems of equations (1) and (2) are more complete than those used earlier by other authors since they take into account the changes in the parameters ($\bar{u}^*$, $\nu_{eff}$, $R_w$) along the longitudinal coordinate and also all the terms that characterize the curvature $\delta/R_w$.

**Görtler Critical Number.** In order to determine the conditions for the occurrence of Taylor–Görtler vortices, an eigenvalue problem was solved for the systems of equations (1) and (2) in the absence of perturbing factors under conditions of the laminar mode of flow ($\bar{\nu}_{eff}=1$). The problem was solved by the "shooting" method for system (2) and the finite-difference method for system (1) with zero boundary conditions. For the solution a polynomial profile was adopted for the averaged longitudinal velocity component [4]:

$$\bar{u}^*=f'(\xi)=F(\xi)+\Lambda G(\xi)+\frac{\delta}{R_w}\frac{\Lambda+12}{1-\delta/R_w}G(\xi)\,,\tag{3}$$

where $F(\xi) = 2\xi - 2\xi^3 + \xi^4$; $G(\xi) = \xi(1-\xi)^3/6$.

The transverse velocity component $\bar{v}^*$ was determined from the continuity equation

$$\bar{v}^* = \frac{d\delta}{dx}(\xi f' - f) - \frac{\delta}{u_\infty}\frac{du_\infty}{dx} f.$$

Bearing in mind that for a laminar mode of flow the boundary layer thickness is defined as

$$\delta = mx/\mathrm{Re}_x^{0.5},$$

we eliminate $d\delta/dx$ from the previous expression. This yields

$$\bar{v}^* = \frac{m^2}{2\,\mathrm{Re}}(\xi f' - f) - \frac{\Lambda}{\mathrm{Re}} f.$$

In the case of gradientless flow, which is considered in the present paper, the second term in the expression for $\bar{v}^*$ is equal to zero.

The values of the coefficient $m$ in the formula for the boundary layer thickness are a function of the curvature $\delta/R_w$. To determine this function, we solved numerically the differential equations of a laminar boundary layer; in the range $\delta/R_w = 0-0.1$ the coefficient $m$ is correlated by the following equation:

$$m = \begin{cases} 4.7 - 2.028\,(\delta/R_w)^{0.415} & \text{at} \quad \delta/R_w = 0\ldots0.03, \\ 4.281 - 1.811\,\delta/R_w & \text{at} \quad \delta/R_w = 0.03\ldots1. \end{cases} \tag{4}$$

The solution of the eigenvalue problem by the finite-difference method made it possible to obtain the following equation for the critical Gö number

$$\mathrm{G\ddot{o}}_{cr} = \begin{cases} 22.7 & \text{for} \quad \delta/R_w \le 0.02, \\ 27.76\left(\dfrac{\delta}{R_w}\right)^{0.0484} & \text{for} \quad \delta/R_w > 0.02\ldots0.1. \end{cases}$$

As is seen from this relation, starting from $\delta/R_w = 0.02$, the value of $\mathrm{G\ddot{o}}_{cr}$ is a weak increasing function of the curvature, and at $\delta/R_w = 0.1$ the value of $\mathrm{G\ddot{o}}_{cr}$ exceeds the minimum value by 8% (for $\delta/R_w \le 0.02$). Converted values of $\mathrm{G\ddot{o}}^{**}$ are presented in Fig. 1 for the case $\delta/R_w \le 0.02$.

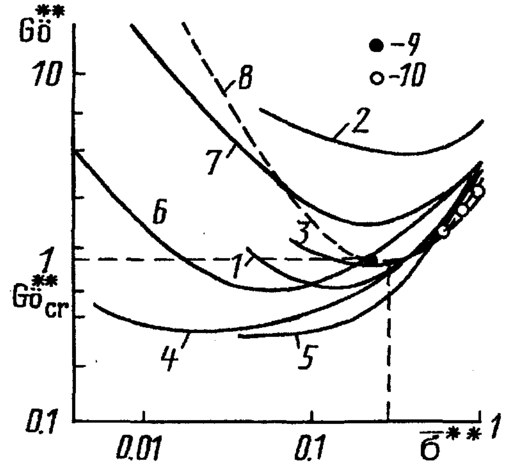
With the use of the "shooting" method, the value of $\mathrm{G\ddot{o}}_{cr}$ depends on the upper boundary of the solution domain. If the upper boundary is the outer edge of the boundary layer, the $\mathrm{G\ddot{o}}_{cr}$ values obtained by the "shooting" method are 1–3% lower than the $\mathrm{G\ddot{o}}_{cr}$ values found by the finite-difference method. When the outer edge is at infinity, the $\mathrm{G\ddot{o}}_{cr}$ value calculated by the "shooting" method is 7–10% lower than that calculated by the finite-difference method.

The accuracy in determining the fulfillment of zero boundary conditions at the outer edge using the "shooting" method is lower than in the finite-difference method. Therefore, the data on $\mathrm{G\ddot{o}}_{cr}$ obtained by the finite-difference method may be considered more soundly based.

The few available experimental data on the neutral stability curve are presented in Fig. 1. These data are close to theoretical curves obtained in the present work and in the work of Aihara.

Figure 2 shows values of the critical wave number corresponding to the appearance of vortices in a boundary layer. From this figure it is seen that calculations made in the present work are in best agreement with experimental data, thus bearing witness to the reliability of the results obtained.

**Perturbing Amplitudes and Surface Friction.** Knowing the eigenvalues of the system of equations (1) or (2) made it possible to solve the equation of motion and obtain the distribution of the perturbing amplitudes.



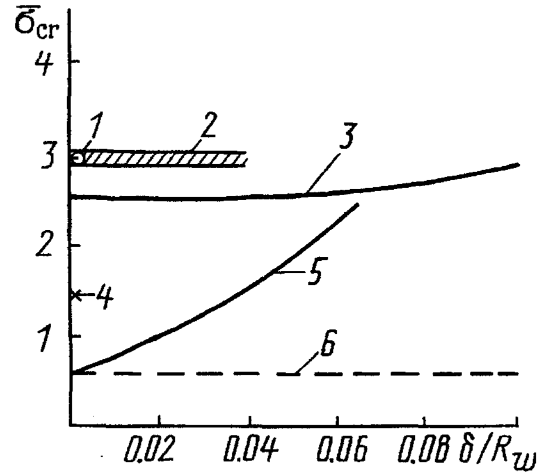




Fig. 1. Stability diagram: 1–8) calculation; 9–10) experiment; 1) Görtler; 2) Meksyn; 3) Aihara; 4) Kahawita; 5) Smith; 6) Herbert ($\delta^{**}/T_w = 6.6\cdot 10^{-6}$); 7) Herbert ($\delta^{**}/R_w = 6.6\cdot 10^{-3}$; 8) results of the present investigation ($\delta/R_w \le 0.02$); 9) Wortmann; 10) Bipps.

Fig. 2. Critical wave number; 1, 2) experiment; 3–6) calculation; 1) Tani; 2) [2]; 3) results of the present investigation; 4) Görtler; 5) Herbert; 6) Smith.

The solution showed that the profiles of the perturbing amplitudes of the longitudinal velocity component normalized to the value of $u_\infty$ are self-similar functions of the Reynolds number provided that Gö = idem. The profile of the longitudinal component of the perturbing amplitude of the velocity (in the region $\xi = 0-1$) is defined by the equations

$$\varphi(\xi) = u_A^* = \begin{cases} 0.7\, c_1 \xi + (0.0258 - 0.007\, c_1)\cdot 10^4\, \xi^2 & \text{when} \quad \xi \to 0, \\ S_1\, \xi \exp(-b\xi) - S_2\, \xi^2 & \text{when} \quad \xi = 0.01\ldots 1.0, \end{cases} \tag{5}$$

where $S_1 = 2.66$; $b = 3.0$; $S_2 = 0.1324$; $c_1 = f''(0)$ is the coefficient in the linear term of polynomial (3).

Profile (5) is in good agreement with the experimental data (Fig. 3). When the second term in Eq. (5) is neglected ($S_2 = 0$), the perturbing velocity decays at infinity, in agreement with the predicted profiles [2].

The other components of the perturbing amplitudes of the velocity are defined by the following equations:

$$v_A^* = -\frac{1}{\mathrm{Re}} S_1\, \xi^2 \exp(-b\xi); \quad w_A^* = \frac{S_1}{\sigma\, \mathrm{Re}}\, \xi\, (2 - b\xi) \exp(-b\xi). \tag{6}$$

From these formulas it is seen that the component $v_A^*$ has negative values over the entire thickness of the boundary layer, while the component $w_A^*$ has positive values in the inner part of the boundary layer and changes its sign when $\xi > 2/b$. Moreover, in contrast to $u_A^*$, these profiles are not self-similar with respect to the Reynolds number (and, consequently, to the Görtler number).

Using distributions (3) and (5), we can find the coefficient of friction ($\mathrm{Re}_x$ = idem)

$$c_f = c_{f_0}\, [1 + 0.7 \cos(\sigma z)], \tag{7}$$

where $c_{f_0}$ is the coefficient of friction without allowance for the effect of Taylor–Görtler vortices.

As is seen from Eq. (7), the linear approximation makes it possible to predict a harmonic change in the coefficient at friction across the surface immersed in the flow. In this case the value of the surface friction, equal to the friction under irrotational flow conditions, is preserved on the average. As indicated above, this is expalined by neglect quadratic terms.

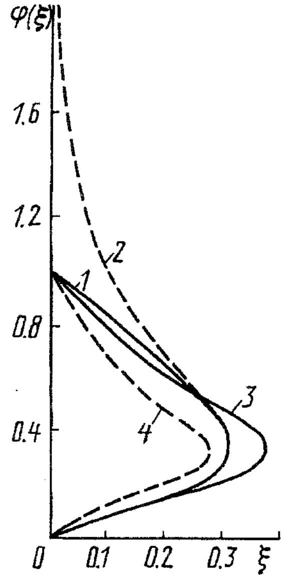


Fig. 3. Profiles of the longitudinal perturbed velocity: 1) calculated from Eq. (5); 2) calculated from Eq. (5) at $S_2 = 0$; 3) experimental data of [1]; 4) experimental data of Yurchenko et al.

**Nonlinear Analysis.** Next, let us analyze the differential equations of motion with account for the quadratic terms in perturbing amplitudes. Following the approach considered in [3], we substitute the first approximation of the velocity into the equation of motion. This yields terms that contain the factor cos $(\sigma z)$, which can be represented in the form of the sum

$$\cos^2(\sigma z) = \frac{1}{2}\,[1 + \cos(2\sigma z)]\,.$$

This equality indicates that the nonlinear terms are related to nonharmonic effects due to harmonic "causes" (the first term of this equation cos $(0\sigma z) = 1$) and nonlinear harmonic effects (the second term). Thus, the perturbing velocities caused by Taylor–Görtler vortices should be sought in the following form

$$u' = u_A \cos(\sigma z) + u_{B_1} \cos(2\sigma z) + u_{B_2}\,;$$

$$v' = v_A \cos(\sigma z) + v_{B_1} \cos(2\sigma z) + v_{B_2}\,;$$

$$w' = w_A \sin(\sigma z) + w_{B_1} \sin(2\sigma z) + w_{B_2}\,. \tag{8}$$

In this case the equation of motion for the perturbing quantitis with allowance for nonlinear terms will be written as

$$\bar{u}\frac{\partial u'}{\partial x} + u'\frac{\partial u'}{\partial x} + u'\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial u'}{\partial y} + v'\frac{\partial \bar{u}}{\partial y} + v'\frac{\partial u'}{\partial y} = \nu\frac{\partial^2 u'}{\partial y^2}\,. \tag{9}$$

Substituting Eq. (8) into Eq. (9) and then equating coefficients of cosines with identical arguments, we obtain equations for linear and nonlinear approximations. The coefficients of cos $(\sigma z)$ refer to the linear approximation, which was considered above. Experimental data show [1, 2] that the coefficients of cos $(2\sigma z)$ are small and can be neglected. Then, the equation consisting of the coefficients that take into account nonlinear terms has the form

$$\frac{1}{2} u_A \frac{\partial u_A}{\partial x} + u_{B_2} \frac{\partial \bar{u}}{\partial x} + \bar{u} \frac{\partial u_{B_2}}{\partial x} + \bar{v} \frac{\partial u_{B_2}}{\partial y} + v_{B_2} \frac{\partial \bar{u}}{\partial y} + \frac{1}{2} v_A \frac{\partial u_A}{\partial y} = \nu \frac{\partial^2 u_{B_2}}{\partial y^2}. \tag{10}$$

To solve Eq. (10), we use for $\bar{u}$ and $\bar{v}$ the Blasius distribution [1]

$$\bar{u} = u_\infty f_B'(\eta); \quad \bar{v} = \frac{1}{2} \sqrt{\left(\frac{\nu u_\infty}{x}\right)} (\eta f_B' - f_B),$$

where $\eta = y \sqrt{u_\infty / v_x}$.

The perturbation of the amplitude of $u_A$ and $v_A$ can be determined from formulas (5) and (6) and that of $u_{B_2}$ and $v_{B_2}$ is prescribed by the following relations:

$$u_{B_2} = u_\infty \psi'(\eta); \quad v_{B_2} = \frac{1}{2} \sqrt{\left(\frac{u_\infty \nu}{x}\right)} (\eta \psi' - \psi).$$

Substituting $\bar{u}$, $\bar{v}$, $u_A$, $v_A$, $u_{B_2}$, and $v_{B_2}$ into Eq. (10), we obtain

$$\frac{d^3\psi}{d\eta^3} + \frac{1}{2} f_B \frac{d^2\psi}{d\eta^2} + \frac{1}{2} \frac{d\psi}{d\eta} \left( \eta \frac{d^2 f_B}{d\eta^2} - \eta \right) + \frac{1}{2} \psi =$$

$$= -\frac{1}{4} \varphi \frac{d\varphi}{d\eta} \eta \left( 1 + \frac{2}{m^2} \right).$$

The right-hand side of this equation is proportional to $\exp(-2b\xi)$, and therefore it is advisable to seek its solution in the form of the series

$$\psi = e^{-2b\xi} \sum_{i=2}^{n} a_i \eta^i. \tag{11}$$

This series begins with the value $i = 2$, since it is necessary to satisfy the condition $u_{B_2} = 0$. It is seen that due to the multiplier $e^{-2b\xi}$, the quantities $u_{B_2}$ and $v_{B_2}$ decay much faster than $u_A$ and $v_A$.

To solve the equation, we used the Galerkin method and represented the Blasius function $f_B'(\eta)$ in the form of a series. The solution gave the following values for the coefficients of series (9):

$$a_2 = \frac{1}{4} \frac{S_1^2}{m} \left( 1 + \frac{2}{m^2} \right) (3 + f_B''(0))^{-1};$$

$$a_3 = \frac{2}{3} \frac{b}{m} a_2; \quad a_4 = -\frac{1}{3} \left( \frac{b}{m} \right)^2 a_2;$$

$$a_5 = -\frac{S_1^2}{240} - a_2 \left( \frac{1}{5} \left( \frac{b}{m} \right)^3 - \frac{f'(0)}{15} \frac{b}{m} - 3 \cdot 10^{-4} \right).$$

Using Eq. (11) with allowance for the expressions for the coefficient $a_2$, we find the tangential shear stress on the wall taking account of the nonlinear effects:

$$c_f = c_{f_0} \left( 1 + \frac{0.7 c_1 \cos(\sigma z)}{m f_B''(0)} + \frac{1}{2} \frac{S_1^2}{m} \frac{(1 + 2/m^2)}{f_B''(0)(3 + f_B''(0))} \right), \tag{12}$$

where $f_B' = 0.332$; $m$ is determined from formula (4); $c_1$ is the first coefficient of polynomial (3).

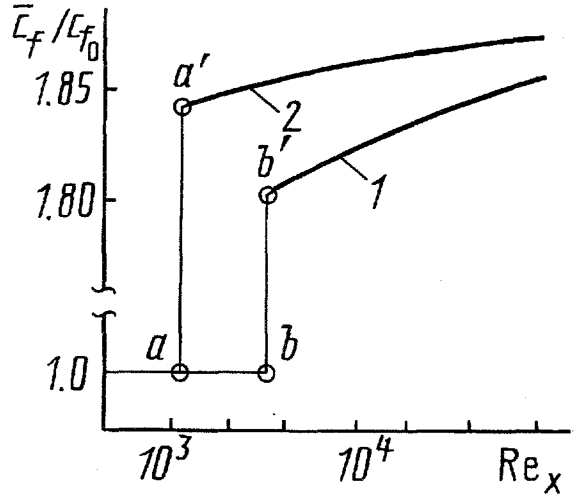



Fig. 4. Variation of the width-averaged coefficient of friction along concave surface: 1) $R_w = 1.5$; 2) 0.5.

As an example, Fig. 4 presents the change in the relative (averaged over the coordinate $z$) coefficient of friction along a concave surface calculated from formula (12). The sharp increase in the coefficients of friction at the points $a$ and $b$ corresponds to the beginning of the appearance of Taylor–Görtler vortices. It is evident that the influence of vortices on friction is very appreciable and attains 80–85%. However, this effect depends weakly on the surface curvature $(\delta/R_w)$.

## NOTATION

$x$, $y$, $z$, Cartesian coordinates; $u$, $v$, $w$, components of the velocity vector in Cartesian coordinates; $\delta$, boundary layer thickness; $R_w$, radius of the surface curvature; $\nu$, kinematic viscosity; $\rho$, density; $\sigma$, wave number; $\xi = y/\delta$; $\bar{\sigma} = \sigma\delta$, $\bar{\nu}_{eff} = \nu_{eff}/\nu$, effective viscosity taking into account molecular and turbulent components; $D = d/d\xi$, $D_\text{п} = D - 1/(R_w/\delta - \xi)$, differential operators; $\text{Re} = u_\infty\delta/\nu$, Reynolds number; $\text{Gö} = \text{Re}(\delta R_w)^{0.5}$, Görtler number; $\text{Gö}^{**}$, Görtler number based on the momentum loss thickness $\delta^{**}$; $\bar{\delta}^{**} = \sigma\delta^{**}$; $f = \delta/u_\infty \partial u/\partial x$, pressure gradient parameter; $\Lambda = \delta^2/\nu du_\infty/dx$, Pohlhausen parameter; $p_A^* = p_A/(\rho u_\infty^2)$; $c_f$, coefficient of friction; $\text{Re}_x = u_\infty x/\nu$. Subscripts: $\infty$, outer edge of the boundary layer; $A$, linear perturbing amplitudes; $B_1$, $B_2$, nonlinear perturbing amplitudes; asterisk, dimensionless velocities referred to $u_\infty$; cr, conditions for the appearance of vortices.

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